ABSENTEEISM, SUBSTITUTES, AND COMPLEMENTS AND THE BANZHAF INDEX

By

Thomas Quint

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YALE UNIVERSITY

Box 208281

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Absenteeism, Substitutes, Complements, and the Banzhaf Index

Thomas Quint
Department of Mathematics, University of Nevada at Reno,
Reno, NV 89557

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Abstract

We consider the voting-with-absenteeism game of Quint–Shubik (2003). In that paper we defined a power index for such games, called the absentee index. Our analysis was based on the theory of the Shapley–Shubik power index (SSPI) for simple games.

In this paper we do an analogous analysis, based on the Banzhaf index instead of the SSPI. The result is a new index, called the absentee Banzhaf index. We provide an axiomatization and multilinear extension formula for this index. Finally, we re-explore Myerson's (1977, 1980) "balanced contributions" property, and the concept of substitutes and complements for simple games (Quint–Shubik 2003), again basing our analysis on the Banzhaf index instead of the SSPI.

Keywords: simple game, Shapley-Shubik power index, Banzhaf index, absenteeism, multilinear extension, balanced contributions, substitute, complement

JEL Classification: C7, C71, D72
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Introduction

In a very recent paper (Quint–Shubik 2003), we explored a simple game model of a decision-making body, in which players had independent random probabilities of being absent. This model we called a voting-with-absenteeism game.

Formally, a voting-with-absenteeism game was defined as a pair $(G;r)$, in which G is an underlying n-player simple game and r_i $(i = 1, ..., n)$ is the probability that player i is present (= not absent) for a vote.

We then defined a power index for such games, called the **absentee index**. Essentially the absentee index is a weighted average of the Shapley-Shubik power indices

(SSPIs) of the 2^n possible subgames of G. The weights are the probabilities that each such subgame will form, given the vector of "presence-probabilities" r.

We then did three things with the absentee index. First, we axiomatized it, much in the manner of Dubey's (1975) axiomatization of the SSPI. Second, we proved a multilinear extension formula for it, thereby generalizing Owen's (1972) formula for the Shapley value in the special case of simple games.

Finally, we provided an alternative proof of Myerson's (1977, 1980) "balanced contributions" property for the SSPI. Take any simple game G, and consider any two players i and j. Then, *the change in i's SSPI caused by absenting player j from G is exactly the same as the change in j's SSPI caused by absenting player i from G.* In fact, we were able to quantify these changes in the players' SSPIs, in terms of the coefficients of the multilinear extension of the game.

Put another way, the balanced contributions property says that i's valuation of j's presence in the game is exactly j's valuation of i's presence. If the two players' valuation of each other's presence is positive, call them **complements**; if negative they are **substitutes**. This suggests a motivation for political alliance or confrontation, based solely on the voting rules of the game.

In the present paper we do the analogous analysis, but based on the Banzhaf index rather than the SSPI. The resulting index for voting-with-absenteeism games we call the absentee Banzhaf index.

First we axiomatize the absentee Banzhaf index. It turns out that the only difference between this axiomatization and that for the absentee index (in Quint–Shubik 2003) is that here we have a much less satisfying Efficiency axiom. This mirrors the comparison between the axiomatizations for the SSPI and the Banzhaf index in Dubey (1975) and Dubey–Shapley (1979).

Next we generalize Owen's (1975) result that the partial derivative of the multilinear extension, evaluated at $(\frac{1}{2}, ..., \frac{1}{2})$, gives the Banzhaf index.

Finally, we re-explore the balanced contributions property. Since the Banzhaf index is a semivalue, it is well-known that it too satisfies this property (Sanchez 1997, Dragan 2000, Carreras-Freixas-Puente 2003). In other words, *the change in i's Banzhaf index caused by absenting player j from G is exactly the same as the change in j's Banzhaf index caused by absenting player i from G.* Again, we quantify the amount of change in the SSPIs in terms of the multilinear extension coefficients. In addition, we define the notions of "Banzhaf substitute" and "Banzhaf complement" in the natural way, and show by example that these may be different from the notions of (SSPI)-complement and (SSPI)-substitute defined in Quint–Shubik (2003).

1 Simple Games and the Banzhaf Index

A **simple game** is a pair (N, V) in which $N = \{1, ..., n\}$ is the **player set** and $V : 2^N \to \{0, 1\}$ is the **characteristic function**. Here 2^N denotes the set of **coalitions**, i.e., the set of subsets of N.

Furthermore, we assume

1) $V(S) \leq V(T)$ whenever $S \subseteq T$ (monotonicity), and

2) $V(\emptyset) = 0$.

Note that we do *not* require that $V(N) = 1$. Hence it is possible for $V(S) = 0$ for all $S \subseteq N$ — we call such a game the **zero game** and denote it by Z^n. Let $\mathcal{G}^n$ be the set of all simple games with n players. Also, let $\tilde{\mathcal{G}}^n = \{G \in \mathcal{G}^n : G \neq Z^n\}$.

Suppose $G = (N, V) \in \mathcal{G}^n$. A coalition S is **winning** if $V(S) = 1$; otherwise (if $V(S) = 0$) it is **losing**. Let W be the set of winning coalitions. A member of W is a **minimal winning coalition** if each of its proper subsets is losing. Let MWC be the set of minimal winning coalitions. The reader will note that any element of $\mathcal{G}^n$ can be defined by either W or MWC in lieu of its characteristic function V. Hence we use the notation "$G = (N, W)$" or "$G = (N, MWC)$" without ambiguity.

Fix $G \in \mathcal{G}^n$. A player i is a **dummy player** (in G) if $V(S \cup \{i\}) = V(S)$ for all $S \in 2^N$; equivalently, i is a dummy player if he is a member of no minimal winning coalition. If $V(\{i\}) = 1$ we call i a **master player**, while if $i \notin S \implies V(S) = 0$ we call i a **veto player**. Finally, a **dictator** is any player who is simultaneously a master player and a veto player.

Suppose G_1 and G_2 are elements of $\mathcal{G}^n$, with characteristic functions V_1 and V_2 respectively. Then the **join** of G_1 and G_2, denoted $G_1 \vee G_2$, is the element of $\mathcal{G}^n$ with characteristic function $\max(V_1, V_2)$. Dually, the **meet** of G_1 and G_2, denoted $G_1 \wedge G_2$, is the element of $\mathcal{G}^n$ with characteristic function $\min(V_1, V_2)$.

Finally, if $G \in \mathcal{G}^n$ and π is a permutation of N, we define $\pi G \in \mathcal{G}^n$ as $(N, \pi V)$, where $\pi V(S) = V(\pi^{-1}(S))\ \forall S \in 2^N$.

Before continuing, let us define the notation "S^c" to mean the complement of S, i.e., the set of players not in S. Also, define S/i to be the set of players in S except for i.

Now suppose $G = (N, W)$ is a simple game. The **Shapley–Shubik power index** (SSPI) (Shapley and Shubik, 1954) is the n-vector ψ, where

$$\psi_i(G) = \frac{1}{n!} \sum_{S \in W : S/i \notin W} (|S| - 1)!(n - |S|)!. \tag{1.1}$$

In a voting system context, $\psi_i(G)$ is a widely regarded measure of the power of player i, in the case where the set of coalitions able to enforce their will is precisely W.

Another such "power index" is the Banzhaf index (Banzhaf 1965, Coleman 1971).[1] Suppose $G = (N, V)$ is a TU game, and suppose $i \in N$. A **swing for *i*** is any $S \in 2^N$ for which a) $i \in S$, b) $V(S) = 1$, and c) $V(S/i) = 0$. In words, a swing is a winning

[1] Banzhaf and Coleman did not define precisely the index defined below: see, for instance, Felsenthal and Machover (1998) for details.

coalition in which i is "critical," in that if he leaves the coalition becomes losing. Intuitively, one may measure i's power via his number of swings η_i. More precisely, the Banzhaf index divides this number by the number of coalitions (2^{n-1}) which contain i; hence it measures the probability that i is critical in a randomly selected coalition that contains him.[2]

Definition: Let $G = (N, V)$ be a simple game. For $i = 1, ..., n$, let $\eta_i(G) = |\{S \in 2^N : S \text{ is a swing for } i\}|$. Then the **Banzhaf index** of G is the n-vector $\beta(G)$ given by $\beta_i = \eta_i/2^{n-1}$.

Remark 1.2: Note that if G is the zero game, then there are no swings for any player, and so $\beta(G) = (0, ..., 0)$.

Next, for any $G \in \mathcal{G}^n$, define $\overline{\eta}(G)$ as $\overline{\eta} = \sum_{i=1}^n \eta_i$. Hence $\overline{\eta}$ denotes the total number of swings in the game.

Theorem 1.3 (Dubey–Shapley 1979): The function $h : \tilde{\mathcal{G}}^n \to \mathbb{R}^n$ satisfies the following axioms:

1) If i is a dummy player in G, then $h_i(G) = 0$. [Dummy]
2) $\sum_{i=1}^n h_i(G) = \overline{\eta}(G)/2^{n-1}$ for any $G \in \tilde{\mathcal{G}}^n$. [Efficiency]
3) If π is a permutation of N and $i \in N$, then $h_{\pi(i)}(\pi G) = h_i(G)$. [Symmetry]
4) For any $G_1, G_2 \in \tilde{\mathcal{G}}^n$,

$$h(G_1) + h(G_2) = h(G_1 \vee G_2) + h(G_1 \wedge G_2). \quad \text{[Additivity]} \tag{1.2}$$

if and only if $h(G) = \beta(G)\ \forall G \in \tilde{\mathcal{G}}^n$.

Remark 1.4: It is easy to see that the result above extends to the case where the domain of h is $\mathcal{G}^n$, not $\tilde{\mathcal{G}}^n$.

Next, define the **multilinear extension** of G (Owen, 1972) as the function $f : \mathbb{R}^n \to \mathbb{R}$ given by[3]

$$f(x_1, ..., x_n) = \sum_{S \in W} \prod_{j \in S} x_j \prod_{j \in S^c} (1 - x_j). \tag{1.3}$$

Theorem 1.5 (Owen, 1975): For any $G = (N, V) \in \mathcal{G}^n$ and any player $i \in N$, $\beta_i(G) = \frac{\partial f}{\partial x_i}(\frac{1}{2}, ..., \frac{1}{2})$.

[2] Of course, assuming that it is equally likely that any such coalition is selected.

[3] Owen actually defined the multilinear extension for any transferable-utility game; the formula given is the restriction to simple games.

2 The Model of Absenteeism and the Absentee Banzhaf Index

We now assume that each player has an independent random probability of being absent from the game. Formally, we define r_i to be the probability that i is "present" in the game (and so $1 - r_i$ is the probability that he is "absent"). A **voting-with-absenteeism** game is thus defined by the quantities $(G; r)$, where $G \in \mathcal{G}^n$ is called the "underlying simple game" and r is an n-vector of probabilities. The class of all n-player voting-with-absenteeism games is denoted by $\mathcal{A}^n$.

Our goal is to define a power index for these games. To this end, suppose we are given a voting-with-absenteeism game $(G; r)$, in which $G = (N, MWC)$. For $S \in 2^N$, let $p_S = \prod_{i \in S} r_i \ \prod_{i \in S^c}(1 - r_i)$. Thus p_S is the probability that the set of voters present is precisely S. Next, define $G^S = (N, MWC^S)$, where $MWC^S = \{T \in MWC : T \subseteq S\}$. G^S is the game that occurs if the set of vote is S; the only minimal winning coalitions are those from the underlying game which are completely composed of members of S. Note that if S contains no minimal winning coalitions, then G^S is the zero-game.

Finally, suppose β (resp., ψ) represents the Banzhaf index (resp., SSPI) operator. We define the **absentee Banzhaf index** ABZ by

$$ABZ(G; r) = \sum_{S \in 2^N} p_S \beta(G^S). \tag{2.1}$$

Also, the **absentee (SSPI) index** ϕ is given by

$$\phi(G; r) = \sum_{S \in 2^N} p_S \psi(G^S). \tag{2.2}$$

Hence the absentee Banzhaf (resp., SSPI) index is simply the weighted average of Banzhaf (resp., SSPI) indices over all possible G^S's, where the weights are the probabilities that the set of players present is S.[4]

Example 2.1: Suppose $N = \{1, 2, 3\}$, $MWC = \{\{1\}, \{23\}\}$, and $r = (\frac{1}{2}, 1, \frac{1}{4})$. Then

$$ABZ(G; r) = p_\emptyset \beta(G^\emptyset) + p_{\{1\}}\beta(G^{\{1\}}) + p_{\{2\}}\beta(G^{\{2\}}) + p_{\{3\}}\beta(G^{\{3\}})$$

[4] Straffin (1988) provides an interesting "probability model" interpretation of the Banzhaf index and SSPI. For the Banzhaf index, every player i is assumed to favor a bill with probability p_i and to be opposed with probability $1 - p_i$, where the value of each p_i $(i = 1, ..., n)$ is independently drawn from the uniform distribution on $[0, 1]$. Then β_i is exactly the probability that i's vote (either for or against) will be pivotal. Alternatively, suppose there is a single $U[0, 1]$ random variable p, which gives the probability that any player favors the bill. Then ψ_i is the probability that i's vote will be pivotal. In either case, the "absentee" version of the index measures the exact same probability, with the proviso that absent players are assumed to be "opposed" with probability 1.

$$
\begin{aligned}
&+ p_{\{12\}}\beta(G^{\{12\}}) + p_{\{13\}}\beta(G^{\{13\}}) + p_{\{23\}}\beta(G^{\{23\}}) + p_N\beta(G) \\
&= 0 * (0,0,0) + 0 * (1,0,0) + \frac{3}{8} * (0,0,0) + 0 * (0,0,0) \\
&+ \frac{3}{8} * (1,0,0) + 0 * (1,0,0) + \frac{1}{8} * \left(0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{8} * \left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}\right) \\
&= \left(\frac{15}{32}, \frac{3}{32}, \frac{3}{32}\right).
\end{aligned}
$$

3 Axioms for the Absentee Banzhaf Index

Consider the following properties, to be considered for any solution concept $h : \mathcal{A}^n \to \mathbb{R}^n$ for voting-with-absenteeism games:

1) NONNEGATIVITY: For any $(G;r) \in \mathcal{A}^n$, $h(G;r) \geq 0$.

2) DUMMY: If i is a dummy player in G, then $h_i(G;r) = 0 \; \forall r$.

3) EFFICIENCY: $\sum_i^n h_i(G;r) = \frac{1}{2^{n-1}} E_r(\overline{\eta})$.

4) SYMMETRY: If π is a permutation of N and $i \in N$, then $h_{\pi(i)}(\pi G; \pi r) = h_i(G;r)$. [$\pi r \in \mathbb{R}^n$ is defined by $(\pi r)_i = r_{\pi^{-1}(i)}$.]

5) ADDITIVITY: For any r, $G_1 \in \mathcal{G}^n$, $G_2 \in \mathcal{G}^n$, $h(G_1;r) + h(G_2;r) = h(G_1 \vee G_2;r) + h(G_1 \wedge G_2;r)$.

6) LINEARITY: For any $i, j \in N$, $h_i(G;r_1,...,r_n)$ is a linear (affine) function of r_j.

One notices that five out the six axioms above are exactly the same as those for the "absentee-SSPI" in Quint-Shubik (2003). The only exception is 3). The new Efficiency axiom states that the sum of all players' payoffs should be $1/2^{n-1}$ times the expected number of swings in the game, where the expectation is taken with respect to the n Bernoulli random variables inherent in r. While having an efficiency axiom dependent on $\overline{\eta}$ is somewhat unsatisfactory, we remark that we are simply emulating the axiom from Theorem 1.3 (from Dubey–Shapley 1979) for the "non-absentee" Banzhaf index.[5]

Theorem 3.1: The unique function $h : \mathcal{A}^n \to \mathbb{R}^n$ which satisfies 1)–6) is $h(G) = ABZ(G) \; \forall G \in \mathcal{G}^n$.

Proof: Almost the same as that for Theorem 3.1 in Quint–Shubik (2003).

[5]For a discussion of Axioms 1), 2), 4), 5), and 6), see Quint–Shubik (2003).

4 A Formula for ABZ

In this section we derive a formula for ABZ. We start with formula (2.1), which is $ABZ_i(G;r) = \sum_{Y\in 2^N} p_Y \beta_i(G^Y)$. We rewrite this as

$$ABZ_i(G;r) = \sum_{Y\in 2^N} p_Y \frac{1}{2^{n-1}} \eta_i(G^Y).$$

To count the number of swings for i in G^Y, consider any swing for i (in G) which is contained in Y. This swing, call it S, will also be a swing in G^Y. But also we can add any elements of Y^c to S and still have a swing in G^Y; hence

$$\begin{aligned}
ABZ_i(G;r) &= \sum_{Y\in 2^N} p_Y \frac{1}{2^{n-1}} \sum_{\substack{S\subseteq W: S\ni i, S/i\notin W,\\ \text{and } S\subseteq Y}} 2^{n-|Y|} \\
&= \sum_{Y\in 2^N} p_Y \frac{1}{2^{|Y|-1}} \sum_{\substack{S\subseteq W: S\ni i, S/i\notin W,\\ \text{and } S\subseteq Y}} 1.
\end{aligned}$$

Now, any S in the rightmost summation above will be counted only whenever the index Y in the leftmost summation contains it. Hence we have

$$\begin{aligned}
ABZ_i(G;r) &= \sum_{\substack{S\subseteq W: S\ni i\\ \text{and } S/i\notin W}} \sum_{Y:Y\supseteq S} p_Y \frac{1}{2^{|Y|-1}} \\
&= \sum_{\substack{S\subseteq W: S\ni i\\ \text{and } S/i\notin W}} \sum_{T\subseteq S^c} p_{S\cup T} \frac{1}{2^{|S|+|T|-1}} \\
&= \sum_{\substack{S\subseteq W: S\ni i\\ \text{and } S/i\notin W}} \sum_{T\subseteq S^c} \prod_{j\in S\cup T} r_j \left(\prod_{j\in (S\cup T)^c} (1-r_j) \right) \frac{1}{2^{|S|+|T|-1}} \\
&= \sum_{\substack{S\subseteq W: S\ni i\\ \text{and } S/i\notin W}} \sum_{T\subseteq S^c} \frac{1}{2^{|S|+|T|-1}} \prod_{j\in S\cup T} r_j \left(\sum_{Q\subseteq (S\cup T)^c} (-1)^{|Q|} \prod_{j\in Q} r_j \right).
\end{aligned}$$

We wish to write the above sum in the form $\sum_S \sum_{X\subseteq S^c} a_X \prod_{j\in S\cup X} r_j$, for some coefficients $\{a_X\}$. To do this, for each X we need to collect all of the "$\prod_{j\in S\cup X} r_j$-terms." There will be $2^{|X|}$ such terms, one for each $T\subseteq X$. For each such T, Q will be equal to X/T, and so a_X will be equal to

$$\sum_{T\subseteq X} (-1)^{|X/T|} \frac{1}{2^{|S|+|T|-1}}.$$

Hence,

$$ABZ_i(G;r) = \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \sum_{X \subseteq S^c} \sum_{T \subseteq X} (-1)^{|X/T|} \frac{1}{2^{|S|+|T|-1}} \prod_{j \in S \cup X} r_j.$$

Next, for each $T \subseteq X$ with $|T| = t$ ($t = 0, ..., |X|$), the number of such T's is $\binom{|X|}{t}$; hence we can rewrite the sum as

$$\begin{aligned}
ABZ_i(G;r) &= \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \sum_{X \subseteq S^c} \sum_{t=0}^{|X|} \binom{|X|}{t} (-1)^{|X|-t} \frac{1}{2^{|S|+t-1}} \prod_{j \in S \cup X} r_j \\
&= \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \sum_{X \subseteq S^c} \sum_{m=0}^{|X|} (-1)^m \binom{|X|}{m} \frac{1}{2^{|S|+|X|-m-1}} \prod_{j \in S \cup X} r_j \\
&= \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \sum_{X \subseteq S^c} \left(\sum_{m=0}^{|X|} (-1)^m \binom{|X|}{m} 2^m \right) \frac{1}{2^{|S|+|X|-1}} \prod_{j \in S \cup X} r_j.
\end{aligned}$$

It is now straightforward to show (using the binomial theorem) that the term in large parentheses is equal to $(-1)^{|X|}$; hence we arrive at the formula

$$ABZ_i(G;r) = \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \sum_{X \subseteq S^c} (-1)^{|X|} \frac{1}{2^{|S|+|X|-1}} \prod_{j \in S \cup X} r_j. \tag{4.1}$$

Remark 4.1: One may compare formula (4.1) with the analogous formula for the absentee (SSPI) index in Quint–Shubik (2003), namely

$$\phi_i(G;r) = \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \sum_{X \subseteq S^c} (-1)^{|X|} \frac{1}{|S| + |X|} \prod_{j \in S \cup X} r_j.$$

Example 4.2: We calculate $ABZ_1(G;r)$ using formula (4.1), in the case where G is the game from Example 2.1 ($N = \{1,2,3\}$ and $MWC = \{\{1\},\{23\}\}$).

In this case $\{S \subseteq W : S \ni 1 \text{ and } S/1 \not\in W\} = \{\{1\},\{12\},\{13\}\}$, and so

$$\begin{aligned}
\phi_1(G;r) = {} & (-1)^0 \frac{1}{2^{1+0-1}} r_1 + (-1)^1 \frac{1}{2^{1+1-1}} r_1 r_2 + (-1)^1 \frac{1}{2^{1+1-1}} r_1 r_3 + (-1)^2 \frac{1}{2^{1+2-1}} r_1 r_2 r_3 \\
& + (-1)^0 \frac{1}{2^{2+0-1}} r_1 r_2 + (-1)^1 \frac{1}{2^{2+1-1}} r_1 r_2 r_3 \\
& + (-1)^0 \frac{1}{2^{2+0-1}} r_1 r_3 + (-1)^1 \frac{1}{2^{2+1-1}} r_1 r_2 r_3 \\
= {} & r_1 - \frac{1}{4} r_1 r_2 r_3.
\end{aligned}$$

If $r = (\frac{1}{2}, 1, \frac{1}{4})$ this is equal to $\frac{15}{32}$, which agrees with our answer from Example 2.1.

5 A Multilinear Extension Formula for ABZ

We now state a generalization of Owen's formula (Theorem 1.5):

Theorem 5.1: Let $(G;r)$ be a voting-with-absenteeism game, and let f be the multilinear extension of the simple game G. Then, for any i, we have

$$ABZ_i(G;r) = r_i \frac{\partial f}{\partial x_i}\left(\frac{1}{2}r_1, ..., \frac{1}{2}r_n\right).$$

Remark 5.2: It is clear that in the case where $r = (1, ..., 1)$ we get Theorem 1.5.

Remark 5.3: Again let us revisit Example 2.1/4.2 (where $N = \{1, 2, 3\}$ and $MWC = \{\{1\}, \{23\}\}$). The multilinear extension is $f(x_1, x_2, x_3) = x_1(1 - x_2)(1 - x_3) + x_1x_2(1 - x_3) + x_1x_3(1 - x_2) + x_2x_3(1 - x_1) + x_1x_2x_3 = x_1 + x_2x_3 - x_1x_2x_3$. So $\frac{\partial f}{\partial x_1} = 1 - x_2x_3$, and $r_1\frac{\partial f}{\partial x_1}(\frac{1}{2}r_1, \frac{1}{2}r_2, \frac{1}{2}r_3) = r_1 - \frac{1}{4}r_1r_2r_3$. This agrees with the formula for $ABZ_1(G;r)$ that we found in Example 4.2.

Proof of Theorem 5.1: Taking the derivative of equation (1.3), we get

$$\frac{\partial f}{\partial x_i} = \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \prod_{j \in S : j \neq i} x_j \prod_{j \in S^c} (1 - x_j).$$

Evaluating this at $x_i = \frac{1}{2}r_i$ $(i = 1, ..., n)$, we get

$$\begin{aligned}
\frac{\partial f}{\partial x_i}\left(\frac{1}{2}r_1, ..., \frac{1}{2}r_n\right) &= \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \left(\prod_{j \in S/i} r_j\right) \frac{1}{2^{|S|-1}} \prod_{j \in S^c}\left(1 - \frac{1}{2}r_j\right) \\
&= \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \left(\prod_{j \in S/i} r_j\right) \frac{1}{2^{|S|-1}} \left(\sum_{X \subseteq S^c} (-1)^{|X|} \left(\prod_{j \in X} r_j\right) \frac{1}{2^{|X|}}\right) \\
&= \sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \sum_{X \subseteq S^c} (-1)^{|X|} \frac{1}{2^{|S|+|X|-1}} \left(\prod_{j \in S \cup X/i} r_j\right).
\end{aligned}$$

Hence $r_i\frac{\partial f}{\partial x_i}(\frac{1}{2}r_1, ..., \frac{1}{2}r_n)$ is equal to

$$\sum_{\substack{S \subseteq W : S \ni i \\ \text{and } S/i \notin W}} \sum_{X \subseteq S^c} (-1)^{|X|} \frac{1}{2^{|S|+|X|-1}} \left(\prod_{j \in S \cup X} r_j\right),$$

which is $ABZ_i(G;r)$ from formula (4.1). ∎

6 Banzhaf Substitutes and Complements

In this section we continue the analysis done in Quint–Shubik (2003), on the subject of substitutes and complements in simple games. Here we do a similar analysis to that in the aforementioned paper, except based on the Banzhaf index instead of the SSPI.

We start by briefly reviewing what we did in Quint-Shubik (2003). Let $G = (N, V)$ be a simple game, and suppose $i, j \in N$. We say that j is **SS complementary for**[6] i if $\phi_i(G; 1) - \phi_i(G; 1_j) < 0$; j is **SS substitutionary for** i if $\phi_i(G; 1) - \phi_i(G; 1_j) > 0$; and j is **SS neutral for** i if $\phi_i(G; 1) - \phi_i(G; 1_j) = 0$.[7] Hence j complementary for i means that i's power (as measured by his SSPI) goes down if j becomes absent (and so j's *presence* in the game is *good* for i). A similar interpretation holds for "j is SS substitutionary for i" and "j is SS neutral for i."

In Myerson (1977, 1980) the author defined his "balanced contributions" property, which implies $\phi_i(G; 1) - \phi_i(G; 1_j) = \phi_j(G; 1) - \phi_j(G; 1_i)$ for any G, i, and j. Hence if i is SS complementary (resp., substitutionary, neutral) for j then j is SS complementary (resp., substitutionary, neutral) for i. This in turn means that any pair of players in a simple game can be classified simply as either complements, substitutes, or neutrals.

In Quint-Shubik (2003), we went one step further, showing that

$$\phi_i(G; 1) - \phi_i(G; 1_j) = \phi_j(G; 1) - \phi_j(G; 1_i) = - \sum_{S \in 2^N : S \ni i,j} a_S \frac{1}{|S|}, \tag{6.1}$$

where a_S ($S \subseteq N$) is the coefficient of $\prod_{k \in S} x_k$in the multilinear extension of G. Hence we "quantify" the amount of the balanced contributions.

We now state and prove the analogous theorem in the Banzhaf case.

Theorem 6.1: Let G be a simple game, and let i and j be any two players in the game. *Suppose further that the multilinear extension of G is given by $f(x_1, ..., x_n) = \sum_{S \in 2^N} a_S \prod_{k \in S} x_k$, for some constants $\{a_S\}$. Then*

$$ABZ_i(G; 1) - ABZ_i(G; 1_j) = ABZ_j(G; 1) - ABZ_j(G; 1_i) = - \sum_{S \in 2^N : S \ni i,j} a_S \frac{1}{2^{|S|-1}}. \tag{6.2}$$

Remark 6.2: The result $ABZ_i(G; 1) - ABZ_i(G; 1_j) = ABZ_j(G; 1) - ABZ_j(G; 1_i)$ was actually first proved by Sanchez (1997), and then by Dragan (2000) and Carreras-

[6] In Quint–Shubik (2003), we used the expression "is complementary for" instead of "is SS complementary for"; here we add the adjective "SS" to emphasize that we are using the SSPI (and not the Banzhaf index!) to measure power.

[7] Here the notation "1" for the value of r means a vector of 1's; the notation "1_k" means a vector of 1's except with a "0" in the kth component.

Freixas-Puente (2003). Each of these authors showed that the balanced contributions property holds for all semivalues.

Proof of Theorem 6.1: Since ABZ_i is a linear function of r_j, the left hand side of (6.2) is equal to $(0-1) * \frac{\partial ABZ_i}{\partial r_j}(1, ..., 1)$. Similarly, since ABZ_j is a linear function of r_i, the right hand side of (6.2) is $-\frac{\partial ABZ_j}{\partial r_1}(1, ..., 1)$. Hence the theorem will be proved if we can show

$$\frac{\partial ABZ_i}{\partial r_j}(1, ..., 1) = \frac{\partial ABZ_j}{\partial r_i}(1, ..., 1) - \sum_{S \in 2^N : S \ni i,j} a_S \frac{1}{2^{|S|-1}}. \tag{6.3}$$

To show (6.3), we recall from Theorem 5.1 that $ABZ_i = r_i \; \frac{\partial f}{\partial x_i}(\frac{1}{2}r_1, ..., \frac{1}{2}r_n)$. Here $\frac{\partial f}{\partial x_i} = \sum_{S \in 2^N : S \ni i} a_S \prod_{k \in S/i} x_k$, so

$$ABZ_i = r_i \; \frac{\partial f}{\partial x_i}\left(\frac{1}{2}r_1, ..., \frac{1}{2}r_n\right) = \sum_{S \in 2^N : S \ni i} a_S \left(\prod_{k \in S} r_k\right) \frac{1}{2^{|S|-1}}.$$

>From here we obtain

$$\frac{\partial ABZ_i}{\partial r_j} = \sum_{S \in 2^N : S \ni i,j} a_S \left(\prod_{k \in S/j} r_k\right) \frac{1}{2^{|S|-1}},$$

from which finally

$$\frac{\partial ABZ_i}{\partial r_j}(1, ..., 1) = \sum_{S \in 2^N : S \ni i,j} a_S \frac{1}{2^{|S|-1}}. \tag{6.4}$$

>From the symmetric nature of the right hand side of (6.4) with regard to the variables i and j, we see that we must also have

$$\frac{\partial ABZ_j}{\partial r_i}(1, ..., 1) = \sum_{S \in 2^N : S \ni i,j} a_S \frac{1}{2^{|S|-1}}. \tag{6.5}$$

But together (6.4) and (6.5) imply (6.3), and so the theorem is proven. ■

Definition 6.3: Let G be a simple game, and let i and j be any two players in the game. Then i and j are

a) **Banzhaf substitutes** if $ABZ_i(G;1) - ABZ_i(G;1_j) = ABZ_j(G;1) - ABZ_j(G;1_i) < 0$;

b) **Banzhaf complements** if $ABZ_i(G;1) - ABZ_i(G;1_j) = ABZ_j(G;1) - ABZ_j(G;1_i) > 0$; and

c) **Banzhaf neutrals** if $ABZ_i(G;1) - ABZ_i(G;1_j) = ABZ_j(G;1) - ABZ_j(G;1_i)$ $= 0$.

Next, Theorem 6.1 (and its "twin" result concerning the SSPI) allows us to *order* the pairwise relationships in the game, from "strongest complements" to "strongest substitutes." This suggests a natural heirarchy of partnership and opposition within a voting game, based solely on the set of winning coalitions of the game.

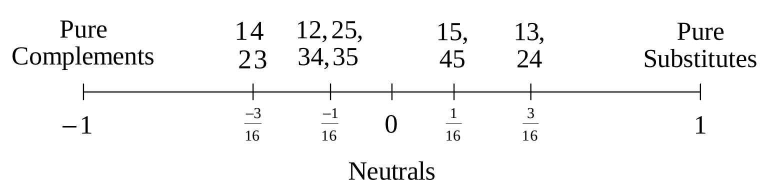
Example 6.4: Suppose $G = (N, MWC)$, where $n = 5$ and $MWC = \{\{12\}, \{234\}, \{235\}, \{14\}\}$. We may verify that

$$\begin{aligned} ABZ(G;1) &= \left(\frac{9}{16}, \frac{7}{16}, \frac{3}{16}, \frac{5}{16}, \frac{1}{16}\right), \\ ABZ(G;1_1) &= \left([0], \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right), \\ ABZ(G;1_2) &= \left(\frac{1}{2}, [0], 0, \frac{1}{2}, 0\right), \\ ABZ(G;1_3) &= \left(\frac{3}{4}, \frac{1}{4}, [0], \frac{1}{4}, 0\right), \\ ABZ(G;1_4) &= \left(\frac{3}{8}, \frac{5}{8}, \frac{1}{8}, [0], \frac{1}{8}\right), \\ ABZ(G;1_5) &= \left(\frac{5}{8}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}, [0]\right). \end{aligned}$$

We get the following values for $ABZ_i(G;1) - ABZ_i(G;1_j)$ $(= ABZ_j(G;1) - ABZ_j(G;1_i))$:

$$\begin{array}{lll} (i,j) = (1,2) : -\frac{1}{16} & (i,j) = (1,3) : +\frac{3}{16} & (i,j) = (1,4) : -\frac{3}{16} \\ (i,j) = (1,5) : +\frac{1}{16} & (i,j) = (2,3) : -\frac{3}{16} & (i,j) = (2,4) : +\frac{3}{16} \\ (i,j) = (2,5) : -\frac{1}{16} & (i,j) = (3,4) : -\frac{1}{16} & (i,j) = (3,5) : -\frac{1}{16} \\ & (i,j) = (4,5) : +\frac{1}{16} & \end{array}$$

Hence we may graph these results on the "complements-substitutes axis" as below:



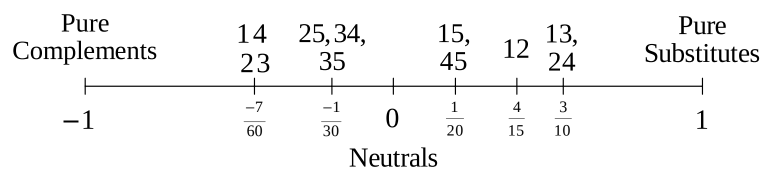
We may also do the same analysis based on the SSPI. The analogous calculations yield

$$\phi(G;1) = \left(\frac{11}{30}, \frac{17}{60}, \frac{7}{60}, \frac{1}{5}, \frac{1}{30}\right),$$

$$\phi(G;1_1) = \left([0], \frac{5}{12}, \frac{5}{12}, \frac{1}{12}, \frac{1}{12}\right),$$
$$\phi(G;1_2) = \left(\frac{1}{2}, [0], 0, \frac{1}{2}, 0\right),$$
$$\phi(G;1_3) = \left(\frac{2}{3}, \frac{1}{6}, [0], \frac{1}{6}, 0\right),$$
$$\phi(G;1_4) = \left(\frac{1}{4}, \frac{7}{12}, \frac{1}{12}, [0], \frac{1}{12}\right),$$
$$\phi(G;1_5) = \left(\frac{5}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{4}, [0]\right).$$

We get the following values for $\phi_i(G;1) - \phi_i(G;1_j)$ $(= ABZ_j(G;1) - ABZ_j(G;1_i))$:

$$\begin{array}{lll} (i,j)=(1,2): -\frac{4}{15} & (i,j)=(1,3): +\frac{3}{10} & (i,j)=(1,4): -\frac{7}{60} \\ (i,j)=(1,5): +\frac{1}{20} & (i,j)=(2,3): -\frac{7}{60} & (i,j)=(2,4): +\frac{3}{10} \\ (i,j)=(2,5): -\frac{1}{30} & (i,j)=(3,4): -\frac{1}{30} & (i,j)=(3,5): -\frac{1}{30} \\ & (i,j)=(4,5): +\frac{1}{20} & \end{array}$$



Comparing these results, we see that for the most part a pair's position on the "Banzhaf complements-substitutes scale" corresponds to its position on the "SS complements-substitutes scale." However, exceptions can and do occur — for instance, here players 1 and 2 are slight complements when judging according to the Banzhaf index, but are strong substitutes when using the SSPI.[8]

In Quint–Shubik (2003), we proved that a) dummy players are SS neutral with all other players; b) in a game with no dummies, a player is a veto player if and only if he is a SS complement with any other player; and c) in a game with no dummies, a master player is a SS substitute with any other player. We now investigate whether these statements still hold in the cases where the adjective "SS" is replaced by "Banzhaf."

[8] Perhaps players 1 and 2 are Banzhaf-complements in this example because they appear together in a minimal winning coalition (MWC), but are SS-substitutes because all the other MWCs contain one or the other, but not both. An interesting problem would be to find mathematical conditions which cause such different categorizations from the Banzhaf and SS indices.

Also, note that in this example 1-2 is the only pair who are substitutes when measured under one index and complements using the other. For a more extreme example, consider the case of a "3-out-of-4 players wins" simple game. In that case *all* pairs of players are substitutes when using the SSPI, while *all* pairs of players are complements when using the Banzhaf index.

First, we note that removing a dummy player from a game does not change any other player's Banzhaf index. Hence *dummy players are Banzhaf neutral with all other players.*

Now consider a game with no dummies. If a veto player absents herself from such a game, the Banzhaf index for any other player moves from something positive to zero. Hence *in a game with no dummies, a veto player is a Banzhaf complement with any other player.* However, note that in a "3-out-of-4 wins" game each player is a Banzhaf complement with all others, yet there are no veto players. Hence, *it is possible that a player who is a Banzhaf complement with every other player is himself not a veto player.*

Finally, suppose i is a master player (in a game with no dummies), and consider any other player j. It is clear that any swing for i in $(G;1)$ is still a swing for i in $(G;1_j)$. Also, if S is a swing for j in $(G;1)$ (such an S exists because j is not a dummy), then $S \cup i$ is a swing for i in $(G;1_j)$, but not in $(G;1)$. These facts together imply that i's Banzhaf index will increase as we move from game $(G;1)$ to game $(G;1_j)$, so i and j are substitutes. Hence, *in a game with no dummies, a master player is a Banzhaf substitute with any other player.*

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